Making Ordinary Mathematics Questions More Inquiry-Based

In this document I show how we used a journal article, which I found relevant and helpful for IO3 in PLATINUM project, to create some new mathematics questions that are more inquire-based.

Journal article

Dorée, S.I. (2017). Turning Routine Exercises Into Activities that Teach Inquiry: A Practical Guide. *PRIMUS*, **27**(2): 179-188.

Abstract:

How can we teach inquiry? In this paper, I offer practical techniques for teaching inquiry effectively using activities built from routine textbook exercises with minimal advanced preparation, including rephrasing exercises as questions, creating activities that inspire students to make conjectures, and asking for counter-examples to reasonable, but false, conjectures.

What makes it interesting for PLATINUM project?

It seems to me a practical approach that most of the university lecturers could easily and willingly adopt, because it is not too time-consuming, and that encourages lectures reflect on the exercises they give tot heir students.

Examples

Example from the Basic Mathematics Course for Biomedical Sciences

Inspired by the paper Marthe Schut and André Heck created new questions on zeros of polynomials of degree greater than 1 and incorporated them in the Basic Mathematics Course for Biomedical Science. They are randomized digital exercises in SOWISO, so only prototypical examples are given below. Students actually get in the course five questions of this kind randomly chosen.

Question 1:

Normally or in the past, the first question would have been "Determine the number of solutions of the following equation." The current question prompt students hopefully more to think about the relation between the equation format and the number of solutions. To quote the author of the cited paper: "Struggling to interpret the question, as opposed to responding to the imperative form of the exercise, leads to deeper conceptual understanding." For example, students may have the alternative conception that that the number of solutions is always equal tot the number of factors in the equation or that equation in the equation always increases the number of solutions. These conceptions are addressed by two types of changes in the equation from part a to part b: (1) squaring one factor and (2) squaring the variable in one factor.

Discussion between CoL members were about the question text (e.g. should we mention that we mean "real solutions" while complex numbers are not treated in this course; should we spend words on what "counting" means: with or without multiplicity) and about the kind of automated feedback that we consider appropriate.

Question 2

The second invites a student to create his/her own example of a polynomial with specified properties. No further restrictions are made than the use of the independent variable t in the entered formula, the degree, and the number of zeros. The computer algebra system Macsyma is used to evaluate the entered formula and to provide feedback: The properties of the entered mathematical expression are compared with the requested properties and automated feedback is provided via statements like "this formula is not a polynomial in t", "the degree of this polynomial is too high/too low", and "this polynomial has too few/too much zeros"

Question 1 (drop-down menu question)

a) How many solutions has an equation that can be written in the following form:
$$(t-4)(t+5)(t-6)=0$$
Answer options: 1|2|3|4|5

b) Now we change the equation as follows:
$$(t-4)^2(t+5)(t-6)=0 \;\; | \;\; (t^2-4)(t+5)(t-6)=0 \;|\; (t-4)(t^2+5)(t-6)=0$$
Is there a change in the number of solutions compared to the previous equation?

Answer options: Yes, there are more solutions | Yes, there are fewer solutions | No, the number of solutions is the same.

Solution text fragments:
"Squaring the term … has no effect on the number of solutions"
"The change from t to t^2 in the term … results in one more|one less solution for the equation"

Question 2 (open question)

Give an example of a polynomial of degree 4 with 3 zeros.

Sample solution text for a request of an example of a 3rd degree polynomial with 2 zeros:
There are several correct answers possible. In the below examples we use the independent variable t and we denote the polynomials as $p(t)$. For of polynomial of degree 3 with two real zeros we can multiply one term of the form $t-a$ by the term t^2. Here, a is a positive real number. Examples of polynomials of degree 3 with two real zeros are $t^2(t-75)$ and 26 $t^2(t-85)$.

Example from the Analysis 2A Course for 1st year mathematics students

At the time of writing the PLATINUM proposal it was expected that PLATINUM member Gerrit Oomens would be the lecturer of this course in the study year 2018/2019 and make it more inquiry-based. His wife had already as student assistant in this course developed digital content on limits of sequences and limits of functions (video clips, course notes, and randomized exercises in SOWISO). But another lecturer was selected for his course and he was reluctant in taking an IBME approach. Main objection was that he felt that the 6-week course was already so filled with mathematical content to be taught that he had no idea to make his lectures inquiry-based, the more that it was the first time that he taught this course and had little time to prepare. We tried to convince him to make some changes in the exercises of the tutorials. Below we give some examples of a tryout. However, we could not convince the lecturer, who did not want to go beyond traditional lecture + tutorial setting with exercises from the book (which he felt comfortable with). Below are some examples of how existing exercises of the first week after the first lecture on section 12 and 14 of the textbook (Kenneth A. Ross, Elementary Analysis, Second Edition) were changed and what was the rationale behind the changes. The lecturer however did not use the exercises as he stuck to the exercises in the book.

Before: Exercise 1. Prove that if $\sum|a_n|$ converges and (b_n) is a bounded sequence, then$\sum a_n\, b_n$ converges.

We reformulate so that it asks students to think about how the convergence of this kind of product series compared to the convergence of the separate series. In order to make them consider this theme more broadly, we add the converse statement and formulate it as an open question.

After: Exercise 1. that if $\sum|a_n|$ be a convergent series and (b_n) is a sequence.

a) If $\sum a_n\, b_n$ converges, does it then hold that (i) $\lim_{n\to\infty} b_n = 0$ or (ii) that the sequence (b_n) is bounded?
b) Use the Cauchy criterion the investigate if $\sum a_n\, b_n$ converges under either assumption (i) or (ii).

Before: Exercise 2. Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

This is a fairly difficult exercise for students unfamiliar with the subject: you need to treat small and large values of n separately. In order to guide students towards this, we start out with $p = 2$ and ask them to consider the magnitude of the squares compared to the original sequence. Once they understand this case, they can generalize towards arbitrary p, also considering $p < 1$.

After: Exercise 2. Let $\sum a_n$ be a convergent series of nonnegative numbers.

a) For how many values of n can we have $a_n^2 > a_n$?
b) Show that $\sum a_n$ converges as well.
c) What can you say about $\sum a_n^p$ for $p \in (0, \infty)$?

Before: Exercise 3. Let (a_n) be a sequence such that $\liminf |a_n| = 0$. Prove that there is a subsequence $\left(a_{n_k}\right)_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

This asks students to reason about two difficult concepts from a previous course: subsequences and limit inferior. We shift the focus to understanding subsequence in this context, switching to a normal limit. Again we also include a converse statement, and ask for an example.

After: Exercise 3. In this exercise we consider sequences (a_n) with $\lim_{n\to\infty} a_n = 0$.

a) Suppose that $\sum_{k=1}^{\infty} a_n$converges and that $a_n \geq 0$. Does $\sum_{k=1}^{\infty} a_{n_k}$ then also converge for any sub-sequence $\left(a_{n_k}\right)$ of (a_n)?
b) Give an example of a sequence (a_n) with limit 0 for which $\sum a_n$ diverges, but such that there exists a subsequence $\left(a_{n_k}\right)$ such that $\sum a_{n_k}$ converges.
c) Prove that for every sequence (a_n) with limit 0 such a subsequence can be found.

Finally, we add a new exercise where students are asked to consider next week's subject: alternating series. The aim here is to make students see the limitations of this week's methods for such series and think about ways their convergence can be investigated.

After: Exercise 4. Consider $\sum \frac{(-1)^n}{n}$.

a) Do the tests from paragraph 14 give any information on this series? Why/why not?
b) Give the first few terms and group them in pairs. What do you observe?
c) Determine if the series converges.