Problems on application of Existence and Uniqueness Theorem

1. (a) Verify that
$$y(x) = \frac{2}{x} - \frac{C_1}{x^2}$$
is the general solution of a differential equation
$$x^2 y' + 2xy = 2.$$
(b) Show that both initial conditions $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solution. Does this fact violate the Existence and Uniqueness Theorem (EUT)? Explain your answer.

2. (a) Verify that
$$y(x) = C_1 + C_2 x^2$$
is the general solution of a differential equation
$$xy'' - y' = 0. \tag{1}$$
(b) Explain why there exists no particular solution of equation (1) satisfying initial conditions
$$y(0) = 0, \qquad y'(0) = 1.$$
(c) Suggest different initial conditions for this differential equation so that there will exist exactly one particular solution of a new initial value problem. Motivate your choice.

3. (a) Create a third order linear differential equation for which solution exists and is unique. Explain your answer.

   (b) Create a first order nonlinear differential equation for which solution exists and is unique. Explain your answer.

4. The coefficient $p(x) = 2/x$ in a linear differential equation
$$xy' + 2y = 18x^4$$
is discontinuous at $x = 0$.

   (a) According to the EUT will a solution satisfying the initial condition $y(0) = 0$ exist or not?

(b) How does your answer to part (a) agree with the fact that $y = 3x^4$ is the exact solution of the initial value problem

$$xy' + 2y = 18x^4, \qquad y(0) = 0?$$

Explain.

5. (a) Verify that the initial value problem

$$x^2y'' - 6y = 0,$$
$$y(0) = 0, \qquad y'(0) = 0,$$

has infinitely many solutions of the form

$$y(x) = Cx^3.$$

(b) Does this fact violate the EUT? Explain your answer.

6. Decide whether the following statements are true or false. Explain your reasoning.

(a) Solution curves to a differential equation never intersect.

(b) Differential equation

$$(y')^2 = y$$

has a unique solution satisfying initial condition $y(0) = 0$.

(c) Solution of a first order differential equation always exists but may not be unique.