Monotonicity of a function in Calculus

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A. Information for lecturers

Unit description

- **Short description of the unit:** students are invited to proceed several activities that should help them to understand the relationship between the first derivative and monotonicity. Particular tasks are described on the work sheet and students can work in groups or individually. It is up to students whether they decide to fill in the digital version of the worksheet or they wish to receive a hard copy to work with
- **Expected prior knowledge of students:** elementary real functions of one variable and its properties, ability to compute a derivative of a given real function, tangent line and its slope (understanding the relationship between the slope of the tangent line and the function derivative at the given point); practical experience with Geogebra or any other application for plotting graphs is helpful, but not necessary
- **The course and context in which it has been used in HE practice**: the unit was used during the seminar to the course Mathematical Analysis 1 and served as the first example to demonstrate application of the real function's first derivative; students were supposed to know the definition of the first derivative, its geometrical interpretation and to compute the derivative of any real function of one variable (general or at the given point);
- **Estimated duration:** 20-30 minutes depending how much time one wants to give students for the final discussion and if participants are experienced users of Geogebra or they need some time to orientate themselves in the interface of the tool

Learning objectives

Performing activities of the unit enables students to

- recall the knowledge and skills they have received before and use it during performing activities described on the work sheet;
- make observations, formulate findings and justify them in a smaller group or during the whole class discussion to identify the relationship between the first derivative and monotonicity including local extrema;
- get to know/remind the main features of Geogebra, the tool to plot the graph of the function and interpret the visualized data (this activity could be a starting point to get to know the tool).

IBME character

Students execute particular tasks and formulate findings and conclusions. The activity could be characterized as structured inquiry. By questions and tasks given on the work sheet, they should realize the main features of the relationship between monotonicity and first derivative of the function. The final question is open and should provoke students to discuss their findings without any limitation. The teacher should only moderate the discussion and give the final summary of all the findings which seem relevant to the topic.

Mathematical content

The unit is meant as an activity to introduce students with the method to search for the intervals of monotonicity and points of local extrema. When working with work sheets and Geogebra students execute particular tasks or use the knowledge they are supposed to be familiar with:

- know the meaning of the function increasing or decreasing at some point (interval);
- find the first derivative of the function;
- be aware of the relationship between the slope of the tangent line and the first derivative at the given point;
- compute the value of the function at given points;
- understand the geometrical interpretation of the function value sign at the given point or interval.

Previous knowledge of these concepts and skills can help them to understand better and faster the main results of the inquiry.

The function on the worksheet is a polynomial defined for all the real numbers and it is not complicated to determine for which intervals the function is increasing/decreasing and which stationary points are local minima or maxima. Performing the tasks gives students an idea how the method works. But this unit should not stand as the only example how we can investigate the monotonicity of the function. That's why the final question is open and gives students or a teacher possibility to pose difficult questions. This unit needs to be supplied by another examples of functions more complicated, in which students have to think about points for which the function is not defined or the first derivative doesn't exist or the first derivative doesn't change the sign despite the fact it is equal to zero.

Technological pedagogical content knowledge

This activity can give students first-hand experience with Geogebra. Students can learn to write the function formula on the keyboard (virtual or physical), input more than one formula, use Geogebra to compute the value of some expression or plot the graph of the function. If students have worked with Geogebra before, they could focus more on interpretation of visualized data to find answers/solutions to the questions/tasks written on the work sheet.

Learning path

The tasks a) to g) on the work sheet are designed to use the knowledge and skills students should have from the previous classes. These tasks are closed and provoke students to do observations, compare results of the computations and pose questions. Below you'll find students activities within the unit.

a) Plotting the function in Geogebra – preparatory task to visualize the graph, students may use virtual keyboard of the application or physical keyboard of their device to input the function formula
b) Specifying intervals of monotonicity from the graph – students use their work sheets and fill in the table with predefined intervals. They trace the function graph visualized in Geogebra and determine for each interval whether the curve is increasing or decreasing. They may have ask questions concerning end points.

c) Computation of the function first derivative – this activity could be done differently, students can compute the first derivative manually and write the result on the worksheet or they can use the functionality of Geogebra and let it compute by this tool. It is up to them.
d) Determining the slope of the tanget lines for given points – again, students can choose whether to establish given points to the formula and compute the slopes manually on the paper or let Geogebra to compute values instead of them. The table on the worksheet is designed to help students to realize how they can find the slope of the tangent line in given points.
e) Plotting the graph of the function first derivative in Geogebra – students create the graph of the first derivative and can compate it with the graph of the function.
f) Specifying intervals on which the first derivative is positive/negative – students use the graph created during the previous activity, determine intervals where this graph is above/under axis x and summarize recognized information to the table on the work sheet.
g) Computation of the first derivative values for given points – all the three points are stationary, so the result should be zero in all the cases.
h) Making conclusions, posing questions

After completion of the unit, the tables of the tasks b) and f) should be filled so they have the same data. This fact should provoke students to think about the relationship between monotonicity and first derivative. They should realize it is connected with the slope of the tangent line and understand why they later on start to solve the equation $f'(x) = 0$, searching stationary points and investigating intervals of monotonicity.

Lecturers' experiences in HE classroom practice

The worksheet took us 30 minutes at all. Some students worked individually, some made groups. All the seminars were held in the room not equipped by computers so we inform students beforehand they are supposed to use their laptops/tablets/smartphones. Some of them have difficulties to use Geogebra as it was their first experience with the tool. The mainly struggle to input the data or to display parts of canvas especially when using smaller smartphones. We were also prepared to give them paper sheets with graphs of the function and its first derivative in case of problems with Geogebra. Few people asked us for that but most of them worked with Geogebra.
We recognized the unit was successful. Students became so engaged by inquiry we didn't want to interrupt them. The first student finished all the tasks after 11 minutes, the last one after 22 minutes. We then spent few minutes discussing. A lot of students were active during the discussion and came to the right conclusions without our help. After the completion of the unit we selected another function and asked them to find its intervals of monotonicity and local extrema. They worked independently without our help and succeeded.

Students with identified needs

We recommend to provide work sheets ahead of time in an electronic editable format, so students with print disabilities can work with them on the laptop/tablet or another computer-based device. It is better to give students a choice to work in groups or

individually as some of them can prefer to discuss their observations and findings, but another are glad to work alone without frequent interruptions by others.
Students may have difficulties to perform the final task, to formulate conclusions of their inquiry as it is a very open question. We recommend the teacher to transfer this activity into the brainstorming discussion and let the students formulate their findings without any formal evaluation. In such a case students won't be afraid to come with suggestions and questions not easy to reply but reaching the challenging situations we described in the previous chapter.
Blind and nearly blind students are not able to follow graphs visualized by Geogebra. If such students are present, a teacher should ask in advance for an individual adjustment based on transforming graphs into tactile graphics. In such a case we recommend to reserve more time for the activity as exploring tactile graphics is more time-consuming in comparison with following figures by sight.

B. Student learning activities

Worksheet – monotonicity of a function

Consider the function: $f(x) = \frac{1}{4}x^4 - x^3 - 2x^2 + 3$.

a) make the graph of the function using the Geogebra application.
b) specify intervals on which the function increases or decreases; write it in the following table:

	$(-\infty,-1\rangle$	$\langle -1,0\rangle$	$\langle 0,4\rangle$	$\langle 4,\infty)$
$f(x) = \frac{1}{4}x^4 - x^3 - 2x^2 + 3$				

c) compute the first derivative of the function.

d) determine the slope of the tangent line to the graph of the function intersecting in the following points; write it in the following table:

x_0	-4	$-0,5$	3	5
$f'(x_0)$				

e) use Geogebra to create the graph of the function's first derivative
f) specify intervals on which the first derivative is positive or negative; write it in the following table

interval				
$f'(x)$				

g) compute: $f'(-1)$, $f'(0)$, $f'(4)$

Final task: make conclusions from your inquiry.