

Unit design: Slope fields and solution curves

In this note we give a short description about what motivated us to pay more attention to slope field in lessons than we did before in our lessons. We have a look at results from educational research literature and compare them with our own classroom experiences. This automatically brings us to the design of the student activities for both in lecture and tutorial.

Motivation

In the past, in a basic mathematics course for 1st year psychobiology students, we paid little attention to the slope field of an first-order ordinary differential equation. Instead we focused on the procedural solving of ODEs, that is, on the finding of an implicit or explicit algebraic expression of a solution via standard methods like separation of variables. The following text fragment shows that in the course notes we even discuss the general case before looking at particular example. This is a typical case of overestimating the mathematical abilities of the students and ignoring their potential conceptual difficulties with slope fields.

A dynamic system of first order that can be written in the form

$$\frac{dy}{dt} = \varphi(t, y)$$

can be "solved" in the following geometrical manner.

Take a point (a, b) and suppose that there is a solution $y(t)$ of the ODE whose graph goes through (a, b) . For this function y holds

$$y(a) = b$$

and

$$y'(a) = \frac{dy}{dt}(a) = \varphi(a, y(a)) = \varphi(a, b)$$

The tangent line of the graph of y at the point a, b is completely determined by these two numbers (see the above section). The equation of the tangent line is namely

$$y = b + \varphi(a, b) \cdot (t - a).$$

The function y itself is unknown, but the tangent line of y at point (a, b) is known. If you now draw at the point (a, b) a small piece of that tangent line, then it will resemble the graph of the solution whose graph goes through (a, b) . Such a line piece is called a **lineal element** at a point in the t, y plane. For many different points (a, b) in t, y plane one can draw lineal elements. A drawing with many lineal elements is called a **slope field** or **direction field** of the ODE. Usually one uses a regular grid of points, but this is not necessary. By drawing a smooth curve that is tangent at any point on the lineal elements we get a so-called **integral curve**, the graph a solution of the ODE. Such a curve is also called a **solution curve**.

It is not surprising that we experienced in tutorials (this course has no lectures anymore) that many a student had difficulties with understanding how a slope field is constructed for a given ODE and what one could do with it (except just 'going with the flow of lineal elements').

For the basic mathematics course for 1st year psychobiology students it was not considered a big issue that slope fields was a topic of minor importance. We could live with the course contents and just try to find time to gradually improve it.

But for the new course for 1st year students in biomedical sciences, the situation has changed completely. This course can be considered as an introduction into system biology and therefore dynamic systems is core business. Because we do not treat the topic of integration, we must rely on computer simulations to let students explore biomedical examples of processes of change and to let them find out how mathematics can help understand the nature of equilibria. Slope fields for first order differential equations and direction fields for systems of differential equation are important tools, also when applied in a qualitative sense.

Some results from educational research literature

Differential equations are a subject at tertiary level and a consequence is that less research has been done about than for mathematical subjects at primary and secondary level. Advances in technology, together with an increased interest in dynamic systems and in particular in nonlinear systems, has changed first courses in differential equations, especially for non-mathematics students.

Chris Rasmussen and Karen Whitehead (2003) reported in their MAA Research Sampler about learning and teaching ordinary differential equations that, in order to be successful in this area, learners must be able to

- move flexibly between algebraic, graphical, and numerical representations;
- make interpretations from the various presentation of situations being modelled;
- make warranted predictions about the long-term behaviour of solutions.

Student difficulties with ODEs, such as alternative ideas and conceptual gaps, can often be found in relation to the above wish list of abilities. Rasmussen (2001) found, for example, that behind students' correct answers there often lay an incorrect conception of equilibrium solution. Student may get the impression for examples of autonomous ODEs that an equilibrium solution exists when the equation is zero. Then they incorrectly conclude that the ODEs $\frac{dy}{dt} = y - t$ and $\frac{dy}{dt} = t + 1$ have steady states $y = t$ and $t = -1$, respectively. A typical case of overgeneralization. One possible explanation, brought up by Rasmussen, involves the difficulty of conceptualizing a solution as a function that satisfies the differential equation. Students may consider equilibrium solution as points where the derivative is zero, rather than as constant functions that satisfy the differential equation, simply because they associate the derivative with the slope of the tangent line at a point.

Rasmussen also found that although students were able to carry out a stability analysis following some graphical approach this did not automatically link their sketches with solutions curves. In the MMA Research Sampler the finding is generalized to the statement "graphical and qualitative approaches do not *automatically* translate into conceptual understanding". The authors' advice is to give students ample opportunities to expand and mathematically defend their conclusions in their work. Rasmussen is in favour of the Realistic Mathematics Education (RME) framework for realizing this. Rasmussen and Kwon (2007) describe an inquiry-

oriented approach to undergraduate mathematics that they have used in their Inquiry Oriented Differential Equations project and this is based on RME.

With regards to slope fields, researchers are cited who found evidence of weak student images of the Euler's methods for finding a numerical solution of an ODE. Michelle Artigue (1992), for example, found that the students' mental image of Euler's method is similar to that of a semi-circle inscribed with a series of line elements. Rasmussen found that students sometimes thought that numerical solutions 'track' the exact solution by using the slope of the exact solution at the start of each next step.

Sketching solutions curves in the slope field is also troubled by overgeneralization: solutions that tend to converge continue converging. We have also noticed in our hands-on activities that students when drawing a solution curve thought that once it starts moving on a nullcline in some direction, it will continue in that same direction, whereas it may just go in the direction to an attractive steady state and come to a standstill near that equilibrium. This misconception may exist for other isoclines as well.

Saber Habre (2000) is cited in the MMA Research Sampler for his observation that students being exposed for years of schooling to the importance of analytical and algebraic settings have some difficulties in accepting the same status for graphical and numerical settings as ways to understand solutions of differential equations. He also points at the students' difficulty of understanding the meaning of solution of a differential equation. Raychaudhuri (2008) argues that the definition of solution entails the context of differential equations (versus the context of solution for an algebraic equation), a mathematical entity (in this case as function), a process that specifies the desired property (in this case the function must satisfy the differential equation), and an object with an implied or implicit process that generates the entity (e.g. separation of variables or an integrating factor). In one of the lectures we were also faced with a student who dared to ask the question after being exposed to three worked-out examples of solving an ODE: "but sir, what do you actually mean by saying 'solving a differential equation'? I still don't get it".

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Relevant student activities prior to the study of slope field

We were aware from the start of the basic mathematics course for 1st year students in biomedical sciences that a slope field of an ODE would be one of the key concepts in our introductory course on dynamics systems. We also had already decided that students would use the R environment to apply the concept in real applications because there was no time for learning analytical methods such as separation of variables because we would not deal with integration of functions. This would give students with a Mathematics A background from secondary school instead of a Mathematics B background a less disadvantageous starting point in the course. Before going into details of the design of the unit on slope fields, we describe the prior teaching and learning in the course that is relevant for understanding slope fields.

The introduction to differential equations was envisioned to happen in steps: first familiarizing students with the concept of differential equation through simple models like exponential growth and limited exponential growth. This would allow students to see that properties of functions can also be expressed in terms of equations that involve the derivative of the function. Students could in this way learn a bit of the terminology of differential equations, such as initial value problem and what solving an ODE means, before they would get a more formal introduction. One could say that we use small steps and repetition to familiarize students with the topic before we would teach it in a more formal way and introduce the slope field as well.

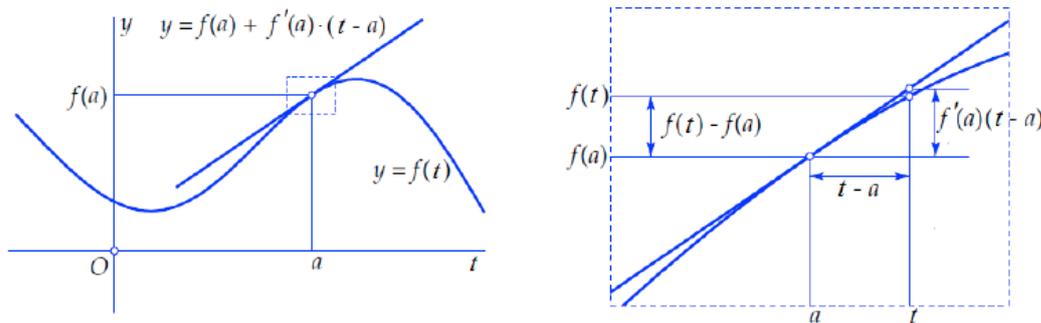
Being aware that slope is not an easy mathematical concept, but has many facets, and realizing that students must have a good conceptual understanding of it before they can grasp a slope field we spent considerable time and effort (one week) to let students deal with derivatives. This subject is taught at secondary school, but we put much more emphasis on (i) the use of a tangent line at a point of the graph of a function as an approximation of the graph of the function in the neighbourhood of that point; and (ii) the notion of difference quotient. We also tried to let student be more aware of the connections between various aspects of differentiation and derivatives.

What did that mean for the lecture? Our planning was to ask first the following three questions, prepared on the presentation sheet, in order to start a classroom discussion about what students already know:

1. What is the derivative of $y = 2t^3 - t + 1$?
2. When y is a function of x , explain in words what the following means: $\frac{dy}{dx} = 5$ for $x = 10$.
3. The derivative of the function f is given by $f'(t) = t^2 - 5t + 3$. What is the slope of the tangent line of f in $t = 1$.

These questions and the discussion were followed by looking more closely at slope, difference quotient (average change), differential quotient (momentary change, as a limit process), and tangent line (as linear approximation of the function near a point). We discuss thoroughly the

following picture and its relation to the algebraic expression $f(t) - f(a) \approx f'(a) \cdot (t - a)$, which can be rewritten as in the form of an equation of a straight line $f(t) \approx f(a) + f'(a) \cdot (t - a)$



We stress the importance of realizing that the graph of a decent function f can be approximated in the neighbourhood of any point $t = a$ by a straight line with slope $f'(a)$. You only have to zoom in enough at the point to see it. This emphasis we feel is needed for good understanding of the slope field of an ODE and its usefulness.

The lecture about differentiation and derivatives continues with discussing the rules of differentiation, derivatives of standard mathematical functions (powers, exponential functions and logarithms; no trigonometric function because we do not use them anywhere in the course as students with mathematics A background know far less about them as students with mathematics B background), higher derivatives, and the uses of derivatives to determine maxima, minima and inflection points. The last part of the lecture is devoted to an inquiry-based approach to let students discover ways to numerically approximate a slope at some point. The task given is:

Given are the following values of a function $y(t)$ in the neighbourhood of $t = 1$:

t	0.7	0.8	1.0	1.1	1.2
$y(t)$	0.741	0.819	1.000	1.105	1.221

What is the best approximation of $y'(1)$? (exact answer = 1 because $y(t) = e^{t-1}$)

Try several methods and compare the results with each other

We discuss this task and its outcomes in class elsewhere. The lesson ends with an application of numerical differentiation to estimate parameters in the logistic growth model of child weight growth by carrying out parabolic regression between the numerical derivative of weight plotted against weight (at certain age) on the basis of real data from a Dutch growth study, and its implementation in the R environment. In the tutorial students are supposed to explore numerical differentiation more deeply, using the R environment as tool to experiment with a noisy signal.

In the lecture and the course notes about differentiation and derivatives, we consider slope as algebraic ratio (difference quotient), a parametric coefficient (in the equation of a straight line), functional property (rate of change between two variables), and as a calculus concept (related to the derivative of a function), and we discuss the link between these aspects of slope and try to better balance the emphasis on these aspects.

For understanding the slope field of a differential equation and its purpose, students must already have a sound idea about what solving a differential equation means. In a lecture about unlimited growth (linear, quadratic and exponential growth model) we spend fair time on looking at what a differential equation actually is. Students already know algebraic equations such as $x^2 = 2$ and what the two solutions are, but they may not realize that one can look at the problem of solving the equation as finding a yet unknown number, denoted by x , that has the property that its square equals 2. Thus, the equation specifies a property of a number we are looking for. Students are also familiar with functional relationship like $x^2 + y^2 = 1$ in which you can isolate one variable and then consider the obtained equation as a function definition. Thus one can look at the equation $x^2 + y^2 = 1$ as the property $x^2 + y(x)^2 = 1$ for a yet unknown function $y(x)$. This prompts students to look at a functional relationship as a property of a mathematical function that has not yet been explicitly defined. For differential equations we want student to take the same perspective: $f'(t) = f(t)$ can be considered as a property of a yet unknown function $f(t)$. Solving the equation can then be considered as finding a more explicit way of describing the function. In this example, it can be an algebraic function definition ($f(t)$ as multiple of the equational function e^t), or a graphical representation (found in some mysterious way), or some numerical method to compute function values at any point of interest such that the numerical derivate and the function have the given property, i.e. 'satisfy the ODE'. In the lecture we discuss the notion of 'general solution of an ODE' and 'particular solution of an ODE', and how extra condition can be used to specify a unique solution. The following ODEs (and related initial value problems) are discussed in the following order of appearance: $y'(t) = 0$, $y'(t) = 2$, $y''(t) = 0$, $y'(t) = 2at + b$, $y'(t) = y(t)$, and $y'(t) = r \cdot y(t)$. In the lecture where we discuss the slope field we first repeat the above discussion about what a differential equation actually is and what it means to solve an ODE.

The design of student activities

In our community of learning we discussed our design of student activities before dividing the work amongst each other. The CoL consists of the course coordinator and lecturer André Heck, and the two tutorial leaders, being the PLATINUM member Marthe Schut and the student assistant Ebo Peerbooms (who generously was paid for a work load of 15 days to help design and implement student activities). The main discussion was about how to introduce the slope field to students and what kind of task would be help them better understand the concept. We decided that we would have a hands-on/brains-on session in the lecture, where the lecturer would first introduce the concept as a joint activity with a worksheet at hand, and that students would go through a series of worksheet task that would be discussed in the lecture room. In the tutorial session student would go through similar tasks to become more proficient with slope field and the sketching of solution curves herein. The would also carry out task to match slope fields with algebraic formulations of ODEs.

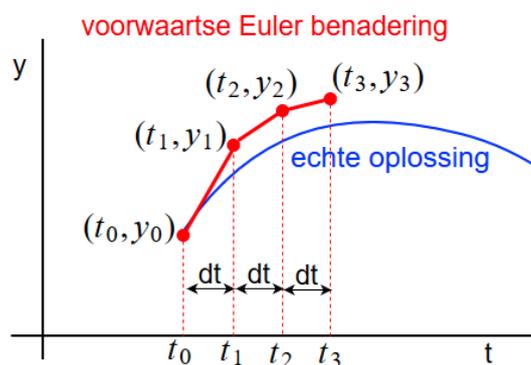
Worksheets for hands-on/brains-on session during the lecture

The first example used in the lecture is the initial value problem $\frac{dy}{dt} = 2 - y, y(0) = 1$. The first questions that the lecturer asks to start group discussion is: What do you know about the solution? Discussion goes on until the students and lecturer arrive together at the conclusion that although the solution y is unknown in $(0,1)$, its tangent line is known and has slope 1. A

small piece of the tangent line (the lineal element) through $(0,1)$ resembles the graph of the solution of the ODE near that point. This raises the question whether you can do this for other points, say (t, y) , in the plane as well. The answer is of course yes and students are invited to draw lineal elements in a prepared worksheet. Thus, in the first worksheet activity they learn how to draw lineal element in the t, y plane.

The next step is to introduce the students to graphical sketching of solutions based on the slope field. The first introductory example chosen is the ODE $\frac{dy}{dt} = 2$, for the reason that all lineal element have the same slope and all known solutions, which students already know to be straight lines, follow the same direction as the lineal elements. The starting question for group discussion is what is known about the lineal elements the given ODE and what is known about the general solution. Finally the slope field with solution curves is shown to the students and further discussed.

As third example we consider the ODE $\frac{dy}{dt} = t$ with quadratic solution curves. The group discussion follows the same pattern as the second example. But this example is also used to start talk about sketching solution curves on the basis of a set of lineal elements. We exemplify the forward Euler method through this example before we discuss the method in more general setting. The picture used in the lecture tries to tackle the misconception of a sequence of line segment connecting point on the exact solution curve



Students are redirected to the first example of the ODE $\frac{dy}{dt} = 2 - y$ and are invited to sketch solution curves in their worksheets. What is a big advantage in this activity is that one can walk around in the lecture room as lecturer and see at a glance in students' work whether they have grasped the concepts and, if not, what obstacles they still encounter.

The next initial value problem is $\frac{dy}{dt} = \frac{y}{2}, y(1) = 1$. Students are invited to sketch the solution curve. This example is chosen to practice more with the slope field of a differential equation about which they already know more and which they can solve algebraically. Another reason for designing this task is that students learn that one can use any condition to move from a general solution to a particular solution and that this does not always have to be a condition of the form $y(0) = \dots$.

Students continue to work with a slope field for the ODE $y' = y \left(1 - \frac{y}{4}\right)$. Students have studied the logistic growth model before and know about its solutions. But using a familiar

example to practice working with slope field and sketching solution curves herein helps students to look at the long-term behaviour of solution curves. The lecturer discussed through this example the concept of equilibrium and the nature of equilibria (attractive/repelling; stable/unstable/semi-stable). The lecturer may even talk about the phase portrait that summarizes the stability analysis, as a kind of appetizer of what is coming soon.

The next example in the hands-on/brains-on session in the lecture is the ODE $\frac{dy}{dt} = t - y$, in which the right-hand side involves both the independent and dependent variable, and for which all solution curves in the long-term approach a straight line solution. In the tutorial session they will elaborate on a similar ODE, here students are only expected to sketch some solution curves and observe the asymptotic behaviour of solutions of the ODE. Students learn that there are more interesting aspects of asymptotic behaviour of solutions than approaching an equilibrium and that slope fields can give you a good first impression. The next example, with ODE $y' = t^2 - y - 2$ is similar, but now the asymptotic behaviour of solutions is that they all tend to approach a quadratic solution curve.

The final worksheet example concerns the ODE $y' = -2t \cdot y$ and helps students to see that solution curves can sometimes be looked upon as a set of curves of the same shape and behaviour.

Worksheets for hands-on/brains-on session during the tutorial

The tutorial also contains hand-on/brains-on parts in a prepared worksheet. Students are asked to draw a slope field of the ODE $y' = y \left(1 - \frac{y}{5}\right)$ describing logistic growth. The reasons for using a similar example as discussed in the lecture are (i) that the student work will reveal whether they have understood what was taught before or that they need more practice; and (ii) that we know that not all students are present at the lectures and may not even had a look at the lecture recordings. For the latter category of students, the slope field is a new subject. Students are invited to study asymptotic behaviour of solutions and the nature of equilibria for this ODE.

The second major example of slope field is in the tutorial worksheet one that belongs to the ODE $y' = 3 - y - t$. Students are invited to draw lineal elements in a diagram and add two solution curves. Both curves approach in the long term a straight line with slope -1. Students are invited to find the equation of this line, but are free to choose the method for accomplishing this: it may be based on the slope field or be done in an algebraic way. The algebraic approach will be discussed anyway in the final assignment in which students use a linear transformation to turn the problem into finding the general solution of a differential equation of restricted exponential growth. Students explore the slope field and the solution curves of the created ODE and relate the equilibrium of this ODE with the straight line describing the asymptotic behaviour of solution in the original problem. In this way, student experience how familiar mathematics is applied to come to grips with more complicated differential equations. This is a form of closed student inquiry: the methods and the results are predetermined, but the students must carry out the involved mathematics and hopefully get the idea how mathematicians 'play' with differential equations or use various modes of reasoning.