

# Matrix Factorization

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## A. Information for lecturers

### Unit description

*Description:* Using appropriately chosen examples, the student is guided to 'discover' the possibility of factoring a matrix as a product of a lower triangular matrix by an upper triangular matrix, what is known as LU factorization. The task is divided into three parts. In the first one, such factorization is possible for the chosen matrix. Students are invited to guess a result. In the second part, the chosen matrix (very similar to the previous one) could not be factored in this way and students were asked to check their initial guess. In the third part, after describing a suitable methodology, students are asked to guess a final result.

The task was implemented using Matlab Live Editor. Both, computer and paper-and-pencil versions were designed.

The task was performed in two classrooms. The students were divided into working groups of 4 students so that there were 8 such groups in one classroom and 10 in the other. Once finished the task, a group discussion was organized. A representative from each group communicated to the other groups the given answer to each of the 3 questions. At the end of this phase, through a debate led by the teacher, the groups assessed, assumed or criticized the responses and opinions of the other groups.

*Student and discipline level:* This unit is concerned with matrix factorization, a topic that would fit in several science degree programmes, but it was designed for use with second year degree students in *Mathematical Engineering*, *Mathematics* and *Mathematics and Statistics*.

*Prior knowledge:* Expected student and lecturer knowledge and skills are

- Familiarity with the Gauss Method.
- Acquaintance with matrix algebra and, specially, with the matrix-matrix product.

*Estimated duration:* Expected time for task during two lectures: 100-120 minutes depending how much time one wants to give students for generating and using ideas and guessing, and on the length and deepness of classroom discussion one prefers or has time for.

### Learning objectives

At completion of the unit, students will be able to

- find that some matrices can be written as the product of two triangular matrices;
- guess that, after permutation of its rows, any matrix admits such a factorization;
- develop investigations (computer or by hand experiments) to inspect and explain what classes of matrices can be factored in this way.
- compute, if possible, the matrices L and U of the LU factorization from Gaussian elimination.
- think more critically about and reflect on mathematical methods and techniques;
- talk about matrix factorization and explore ideas in collaboration with peers;
- share results to others.

## IBME character

The task can be characterized as structured inquiry, because students were invited to guess a result, were asked to check their initial guess and were asked to guess a final result. The procedure was prescribed. It is expected that students develop inquiry abilities such as engagement (through the challenge of guessing), exploration (to refine the answer), explanation (in the post-task discussion).

## Mathematical content

The main content is

- Gauss method.
- Matrix manipulation.
- Lower and upper triangular matrices.
- LU factorization
- Matrix factorization

## Technological pedagogical content knowledge

The experience of several years teaching the subjects related to matrix factorization has led us to verify that it is perceived as unnatural by students. They are able to understand the results but do not link them with their background.

Students are familiar with the Gauss method to solve linear systems. We take advantage of this familiarity to help them discover LU factorization.

In addition, the activity outlines the computation of matrices L and U, which will help students in their effective implementation.

## Learning path

The task consists of three parts that are sequentially delivered to students. Before starting the third part, a remark on storing multipliers is provided.

The activity during the lecture, in which students think about matrix factorization and guess general results from concrete examples, is an interactive discussion. Inviting them to guess a result, the students are expected to be intrigued and tuned in to explore mathematical methods. The goal is that students experience that by talking about mathematics with others, their own thinking becomes deeper and fruitful.

In the table below of student activities within the teaching and learning unit, we typify them as part of the 7E learning cycle of inquiry.

<i>Part</i>	<i>Activity</i>	<i>E-emphasis</i>
1	Apply Gauss method, multiply matrices	Elicit
1	Guess a result	Engage
2	Question about the conjecture	Explore
2	Guess a new result	Engage
3	Find an overall result	Explore
Discussion	Each group makes their responses public	Explain
Discussion	The groups assessed, assumed or criticized the responses and opinions of the other groups.	Elaborate and extend

## Lecturers' experiences in HE classroom practice

The students were very interested in the task. At first, they were concerned and fearful of having to state a result from a single example. Afterward, they felt more comfortable and behaved more naturally.

The interactive discussion in the classroom was fruitful. Several groups of students got closer to the result after knowing the answers of others. However, some of the used arguments were surprising. For example, some group pointed out the symmetry of the matrix of the first example, an irrelevant question in this context. Finally, the teachers guided the discussion in order to clarify and fix ideas.

## Student with special needs

No special resources have been included for students with special needs.

## Assessment

No suggestions.

## Relevance of/to the real world

Matrix factorization is a powerful tool to solve large linear systems. This type of systems is involved in solving a wide range of problems in science and technology.

## B. Student learning activities

The task in the unit is split into three parts. The envisioned student engagement in each activity has already been listed in the description of the learning path in the section *Information for lecturers*.

**Part 1:** Compute the LU factorization of a matrix that does not need permutations in the Gauss method.

*Learning objective:* Figure out the existence of LU factorization for this class of matrices.

*Student activity:* Applying Gaussian elimination, conveniently storing multipliers to obtain matrix L. Compute the product LU. Proposing a conjecture.

*Tool use:* If the task is presented by means of Matlab Live Editor, this tool will be used. Otherwise, paper-and-pencil.

**Part 2:** Attempt to compute the LU factorization of a matrix that need permutations in the Gauss method.

*Learning objective:* Check that, in this case, the permutations can prevent the result.

*Student activity:* Applying Gaussian elimination, storing multipliers to obtain matrix L and permuting rows in U. Computing the product LU. Modifying the conjecture.

*Tool use:* If the task is presented by means of Matlab Live Editor, this tool will be used. Otherwise, paper-and-pencil.

**Part 3:** Computing the  $PA=LU$  factorization of a matrix that need permutations in the Gauss method.

*Learning objective:* Check that permuting rows in L and U leads to the result.

*Student activity:* Applying Gaussian elimination, storing multipliers to obtain matrix L and permuting rows in U. Computing the product LU. Guessing the result.

*Tool use:* If the task is presented by means of Matlab Live Editor, this tool will be used. Otherwise, paper-and-pencil.

### Suggestions for use

Permanent help from the teacher is necessary, especially in the first part.

The three parts should be supplied sequentially, picking up each part when delivering the next, to prevent students from modifying their guesses.

Interactive discussions in the classroom with the students about their answers are very fruitful.

### The task

The complete statement of the task can be found in the following files:

- FactLUInquireEnglish.pdf (annexed and attached)
- FactLUInquireEnglish.mlx (attached)

## First part

We consider a matrix A to which the Gauss method can be applied without permutations. In fact, as we will see, the element that is left on the diagonal at each step (which is called the *pivot*) is the largest among all those that are, in the corresponding column, from the diagonal to the end of the column.

```
A=[8 4 2 0;4 6 1 1; 2 1 4.5 1;0 1 1 2.25]
```

```
A = 4x4
 8.0000    4.0000    2.0000         0
 4.0000    6.0000    1.0000    1.0000
 2.0000    1.0000    4.5000    1.0000
         0    1.0000    1.0000    2.2500
```

We are going to apply the Gauss method to matrix A, **always** choosing the linear combination of rows in which each row is subtracted from the pivot row multiplied by a number (which is called *multiplier*).

We save the initial matrix A in a matrix U in which we are going to make the transformations.

```
U=A
```

```
U = 4x4
 8.0000    4.0000    2.0000         0
 4.0000    6.0000    1.0000    1.0000
 2.0000    1.0000    4.5000    1.0000
         0    1.0000    1.0000    2.2500
```

For the first column, the pivot is 8. Therefore, we will have to subtract from the 2nd, 3rd and 4th rows the 1st row multiplied, respectively, by  $\frac{4}{8} = 0.5$ ,  $\frac{2}{8} = 0.25$  and  $\frac{0}{8} = 0$ . We are going to store these numbers in the matrix L which, initially, is the identity matrix. We will save them in their **first column**, since they are the multipliers used to make zeros in the **first column** of U. We keep the multiplier used to make a zero in the **second row** of U (that is, 0.5) in the **second row** of L, that of the **third** (0.25) in the **third row** of L and that of the **fourth** (0) in the **fourth row** of L.

```
L=eye(4); % Initialize L to identity matrix
L(2,1)=0.5; L(3,1)=0.25; L(4,1)=0;
L
```

```
L = 4x4
 1.0000         0         0         0
 0.5000    1.0000         0         0
 0.2500         0    1.0000         0
         0         0         0    1.0000
```

What does matrix U look like after having made these linear combinations of rows on it?

U=

Once the zeros in the first column have been achieved, the pivot of the second column, that is, the element U (2,2), should be 4.

Calculate the multipliers for the second column, save them in L (3,2) and L (4,2) and make the transformations in matrix U. Then do the same for the third column.

If everything went well, you will have the matrices

$$L = [1 \ 0 \ 0 \ 0; 0.5 \ 1 \ 0 \ 0; 0.25 \ 0 \ 1 \ 0; 0 \ 0.25 \ 0.25 \ 1]$$

L = 4x4

1.0000	0	0	0
0.5000	1.0000	0	0
0.2500	0	1.0000	0
0	0.2500	0.2500	1.0000

and

$$U = [8 \ 4 \ 2 \ 0; 0 \ 4 \ 0 \ 1; 0 \ 0 \ 4 \ 1; 0 \ 0 \ 0 \ 1.75]$$

U = 4x4

8.0000	4.0000	2.0000	0
0	4.0000	0	1.0000
0	0	4.0000	1.0000
0	0	0	1.7500

Calculate L \* U and relate this product to matrix A

**Question 1:** Conjecture a result that accounts for what is described in this first part..

## Second part

We are now going to apply the Gauss method to the matrix shown below. As before, we always choose the linear combination of rows in which the row of the pivot multiplied by the multiplier is subtracted from each row, taking as pivot the largest possible.

$$B = [0 \ 1 \ 1 \ 2.25; 4 \ 6 \ 1 \ 1; 8 \ 4 \ 2 \ 0; 2 \ 1 \ 4.5 \ 1]$$

As in the first part, we save the original matrix in the matrix U in which we are going to make the transformations

$$U=B;$$

To start with, we first swap the first row and the third row obtaining

$$U = \begin{bmatrix} 8.0000 & 4.0000 & 2.0000 & 0 \\ 4.0000 & 6.0000 & 1.0000 & 1.0000 \\ 0 & 1.0000 & 1.0000 & 2.2500 \\ 2.0000 & 1.0000 & 4.5000 & 1.0000 \end{bmatrix}$$

Next, we save the multipliers in the first column of L and make the corresponding linear combinations of rows in U

$$L = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0.5000 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0.2500 & 0 & 0 & 1.0000 \end{bmatrix}$$

$$U = \begin{bmatrix} 8.0000 & 4.0000 & 2.0000 & 0 \\ 0 & 4.0000 & 0 & 1.0000 \\ 0 & 1.0000 & 1.0000 & 2.2500 \\ 0 & 0 & 4.0000 & 1.0000 \end{bmatrix}$$

Looking at the second column of U (the 3 elements from the diagonal down), we already have the largest of them on the diagonal, so we don't have to perform any permutation. Next, we keep the two multipliers in the second column of L and make the linear combinations in U, obtaining

$$L = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0.5000 & 1.0000 & 0 & 0 \\ 0 & 0.2500 & 1.0000 & 0 \\ 0.2500 & 0 & 0 & 1.0000 \end{bmatrix}$$

$$U = \begin{bmatrix} 8 & 4 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

In the third column, we swap the third and fourth rows in U to put on the diagonal the largest element. The matrix U becomes

$$U = \begin{bmatrix} 8 & 4 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

To finish, we have to save the multiplier in L and replace the fourth row of U by the result of it minus the multiplier by the third row of U, obtaining

$$L = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0.5000 & 1.0000 & 0 & 0 \\ 0 & 0.2500 & 1.0000 & 0 \\ 0.2500 & 0 & 0.2500 & 1.0000 \end{bmatrix}$$

$$U = \begin{bmatrix} 8.0000 & 4.0000 & 2.0000 & 0 \\ 0 & 4.0000 & 0 & 1.0000 \\ 0 & 0 & 4.0000 & 1.0000 \\ 0 & 0 & 0 & 1.7500 \end{bmatrix}$$

Calculate  $L * U$  and compare it with B

**Question 2:** Does this lead you to change your conjecture in question 1? Explain the change, if any.

## Remark

Taking into account that to get each zero of the lower triangular part we use a multiplier (that is, there are as many multipliers as zeros) we are now going to repeat the steps of the first part with matrix A, but keeping the multipliers in the place that the zeros would occupy. Thus, in the lower triangular part of U we will have the lower triangular part of L:

```
U = [8.0000    4.0000    2.0000         0
      0.5000    4.0000         0     1.0000
      0.2500         0     4.0000     1.0000
           0     0.2500     0.2500     1.7500];
```

The previous product  $L * U$  can be made in matlab (you can check that the same thing comes out)

```
L=eye(4)+tril(U,-1); % Extract the strictly lower triangular part
                    % and add 1 to diagonal.
U=triu(U);
L*U
```

## Third part

We are going to repeat what was done with matrix B in the second part, now keeping the multipliers in U, as in the observation. We insist that we always choose the linear combination of rows in which the row of the pivot multiplied by the multiplier is subtracted from each row, taking as pivot the large one.

As before, we save the original matrix B in the matrix U in which we make the transformations

```
U = [      0     1.0000     1.0000     2.2500
      4.0000     6.0000     1.0000     1.0000
      8.0000     4.0000     2.0000         0
      2.0000     1.0000     4.5000     1.0000]
```

To start, we swap the first and the third row of U, and then

```
U = [8.0000     4.0000     2.0000         0
      4.0000     6.0000     1.0000     1.0000
           0     1.0000     1.0000     2.2500
      2.0000     1.0000     4.5000     1.0000]
```

We keep, as has been said in the remark, the multipliers in the first column of U:

```
U = [8.0000     4.0000     2.0000         0
      0.5000     6.0000     1.0000     1.0000
           0     1.0000     1.0000     2.2500
      0.2500     1.0000     4.5000     1.0000]
```

and we make the linear combinations of rows in U (only from the second column to the end, of course, so as not to spoil the multipliers),

$$U = \begin{bmatrix} 8.0000 & 4.0000 & 2.0000 & 0 \\ 0.5000 & 4.0000 & 0 & 1.0000 \\ 0 & 1.0000 & 1.0000 & 2.2500 \\ 0.2500 & 0 & 4.0000 & 1.0000 \end{bmatrix}$$

Again, focusing on the second column (the 3 elements from the diagonal down), we observe that we already have the largest of them on the diagonal, so we do not need to perform any permutation. Next, we store the two multipliers in the second column of U (where the zeros would go) and we make the linear combinations, obtaining

$$U = \begin{bmatrix} 8.0000 & 4.0000 & 2.0000 & 0 \\ 0.5000 & 4.0000 & 0 & 1.0000 \\ 0 & 0.2500 & 1.0000 & 2.0000 \\ 0.2500 & 0 & 4.0000 & 1.0000 \end{bmatrix}$$

For the third stage, we interchange the third and fourth rows in U in order to place on the diagonal the largest element of the corresponding piece of the third column. The matrix U becomes

$$U = \begin{bmatrix} 8.0000 & 4.0000 & 2.0000 & 0 \\ 0.5000 & 4.0000 & 0 & 1.0000 \\ 0.2500 & 0 & 4.0000 & 1.0000 \\ 0 & 0.2500 & 1.0000 & 2.0000 \end{bmatrix}$$

It is **very important** to note that **we have exchanged the entire rows** (including the multipliers).

We keep the multiplier in U and we make the linear combination of rows in U, with which we arrive at:

$$U = \begin{bmatrix} 8.0000 & 4.0000 & 2.0000 & 0 \\ 0.5000 & 4.0000 & 0 & 1.0000 \\ 0.2500 & 0 & 4.0000 & 1.0000 \\ 0 & 0.2500 & 0.2500 & 1.7500 \end{bmatrix}$$

If now, as before, we extract the lower triangular part of U in L (with 1's on the diagonal), we leave its upper triangular part in U and multiply them. What is the product  $L * U$  and how is it related to B?

**Question 3:** Can you draw any conclusions about it?