

Limits of curve sketching in school?

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A. Information for lecturers

Unit description

Description: This unit is an exercise which is divided into a mathematical and a didactical subtask. Both (sub)tasks are included in the exercise material, but the focus of this document will be on the mathematical task.

Mathematical task: Students are first expected to determine correctly the number of extrema and inflection points on two graphs, which are given as drawings (no algebraic formulas or numbers are provided). The graphs are specifically chosen to pose a number of challenges for German first-year students of mathematics education. Students are given a chapter on extrema and inflection points from a school-book as reference material. The correct mathematical solution should be discussed with students, before they proceed to the more advanced didactical task.

Didactical task: Students are expected to provide a subject matter analysis of the mathematical problem(s) posed in the mathematical part of the exercise, i.e. they should describe in detail the mathematical topics that could be discussed with the help of the two graphs in a classroom. They should then make a proposal for a classroom discussion that aims to bring up and clarify the mathematical questions raised by the two graphs.

Student and discipline level: The original target group of this unit are German first-year students of mathematics education for upper high school (Gymnasium).

The unit is concerned with:

- the topic of curve sketching as taught in German upper high schools (concepts of local/global extremum and inflection point; graphical differentiation; theorem application: criteria for extrema / inflection points (for polynomials); necessary and sufficient conditions)
- basic topics of undergraduate mathematics practices (strategy of looking up relevant definitions; basics of logic; composite functions)
- didactical phenomena and questions (e.g. concept image / concept definition; different mathematical discourses; educational goals of curve sketching, reflection of schematic application of criteria, institutional conditions of teaching in school)

Prior knowledge: Students are expected to have knowledge in the following areas:

- Mathematics: Concepts of extremum (local and global) and inflection point are familiar (learned in school)
- Mathematics: Strategies for finding extrema and inflection points on graphs using criteria which involve derivatives of the corresponding function are familiar (learned in school)
- Mathematics: Students are currently studying basics of university mathematics, such as: differences between definitions and theorems, basics of logic
- [Mathematics – not mandatory but helpful: Students can differentiate functions graphically (learned in school)]
- [Mathematics – not mandatory but helpful: composite functions]

- Didactics: students should know some basics of didactics of mathematics from university courses, which they can draw on when making suggestions for possible uses of the graphs in class. (No particular piece of knowledge is necessary though.)

Estimated duration: The duration of the entire unit depends immensely on the lecturer's choices and on the students' prior knowledge and chosen strategies.

The minimal time needed to solve the tasks might be 3-4 hours of productive work by students (assuming that students find the correct approach relatively quickly after failing once with a useless approach) and includes: screening a chapter of a schoolbook (approx. 20 pages); finding the correct approach and/or having the correct approach eventually explained by the lecturer; writing a text that explains the mathematical arguments for the correct answer; developing some ideas on how the graphs could be used in a school lesson and explaining them in a text.

Learning objectives

The learning outcomes of this unit depend heavily on the prior knowledge of each student, on her/his familiarity with university mathematics, on the approaches the student chooses to take in order to solve the task and on how the lecturer uses the task in her/his course (e.g.: Is it used as a point of departure for additional activities, like discussions or the introduction of new ideas?).

The knowledge/skills necessarily involved in solving the task (minimal solution) are:

- Consulting and understanding the schoolbook definitions of (local) extremum and inflection point
- Writing a coherent explanation of the mathematical solution
- Writing a text which explains didactical ideas (application of prior knowledge of didactics)

Knowledge/skills that could be reviewed in the context of the task are:

- Applying school-book theorems to find the mathematical solution (necessary/sufficient conditions of extrema and inflection points that draw on derivatives)
- Graphical differentiation
- Inventing a formula for a graph given as drawing
- The concepts of function (the graphs are unusual and have the potential to challenge students' conceptions of functions), extremum, inflection point and point in general

Knowledge/skills that can be learned in solving the task (if students are unfamiliar with these techniques/concepts and depending on interventions by the lecturer) are:

- Logical analysis of mathematical theorems from the school-book (equivalence versus implication / necessary versus sufficient conditions). Paying attention to the logical structure of theorems helps to avoid mistakes.
- In university mathematics, the conditions of theorems need to be checked before applying them, even if this was not done in school. Considering definitions and checking assumptions of theorems is a viable strategy when trying to solve a mathematical problem.
- Composite functions are not several functions but one function.
The (school-book) definitions of extremum and inflection point were created for functions in general (not just polynomials) and therefore can be applied to composite functions.
- Realizing that a straight horizontal line has in every point (except endpoints) a local maximum and minimum.
- Realizing that an s-shaped curve with a straight line in the middle has no inflection points.

- [Didactics:] Discussing the (adequacy of the) image of the motor cyclist on the curvy mountain road, which is used in the schoolbook to introduce inflection points, in view of the given graph (1) from the exercise.

Mathematical and didactical topics that could be discussed more deeply on the basis of the activity or as part of a discussion of the ideas for the use of the two graphs in school:

- Creation of different composite functions that have graphs similar to the given graph drawings and discussion of their properties and differentiability, specifically at “border points”.
- Discussion of the differences between university and school mathematics. Elements of didactical theory such as some concepts from the *Anthropological Theory of the Didactic (ATD)* or the terms *concept image* and *concept definition* of functions could be introduced in order to describe phenomena concerning differences between school-related practices and university mathematics practices.
- Discussion of extreme cases of definition, for example “Does a horizontal line have minima and maxima?” or “Is a straight line also a curve?”.
- Discussion of the concept of a mathematical point vs. everyday conceptions of the “point” where the motorcyclists changes her/his direction.
- Discussion of pictures in school-books and circumstances/contexts in which they might be misleading.
- Discussion of the educational goals connected with curve sketching in school. What criticism exists in mathematics education literature of the topic of curve sketching and what is suggested to be improved about it?
- Under what circumstances and with what educational goals could the mathematical task / the two graphs be employed in a school classroom? (Answering this question is the second didactical task.)

IBME character

For the mathematical task, students are given the exercise material together with a school-book chapter. They are expected to inquire into the topic on their own, on the basis of their prior knowledge and the textbook excerpt.

Therefore, the **student activity** can be regarded as **open inquiry**.

The **task** the students have to solve **has a fixed solution**.

After some time, the student solutions to the mathematical task have to be reviewed by the lecturer. Depending on what misconceptions and errors are found in student solutions, additional activities and discussions could be organized, before letting students proceed to part two of the exercise, the didactical task.

In the didactical task, students are expected to get creative and come up with new ideas, but have to be able to justify their reasoning with didactical arguments.

The **student activity** is again **open inquiry**.

This time, the **task is open-ended** (it has no pre-determined solution), but **task solutions** have to satisfy certain **quality standards**.

Technological pedagogical content knowledge

Students difficulties concerning the mathematical task:

In Germany, curve sketching is typically taught with a strong focus on applying necessary and sufficient criteria via (algebraic) differentiation of functions. Furthermore, student understanding of the concepts of extremum and inflection point is usually dominated by / restricted to the case of polynomials, although the two concepts are introduced in first-year university mathematics for a more general class of functions. Our students have great difficulties to switch from procedural practices to practices such as checking prerequisites of theorems and consulting definitions. When the usual criteria do not work, students engage in a variety of “unproductive” practices, i.e. practices that does not lead to a solution, such as:

- Students interpret the task instruction in a way that excludes composite functions from their considerations, arguing that the two graphs have to be polynomials.
- Students imagine the graphs to represent composite functions, but don't verify the conditions of the school-book theorems (indefinite differentiability) before applying these theorems to the supposed functions.
- Students successfully verify necessary conditions, and then argue that the graphs have no inflection points or extrema, as sufficient conditions don't hold.
- Students can explain why the necessary and sufficient conditions from the school-book are useless for the task, but are unable to find another approach / the solution.

Student difficulties concerning the didactical task:

As didactics of mathematics is a completely new subject for first-year university students in Germany, they have to be enculturated into this subject. Typical student difficulties are linked to quality standards applied to student contributions and ideas in a didactics classroom.

Technology

Apart from the fact that the unit is implemented on ILIAS (like all other tasks of the course), information and communication technology does not play a big role in the solution of the presented task. On the contrary, we think it is very important that the graphs are presented as handwritten sketches that appear in a discussion between two students at school. This, besides other task features, situates the whole task within a school discourse and stresses the relevancy of the raised questions for school. The questions are intended to stimulate discussion about mathematical practices, which makes the use of technology to solve the problem seem unproductive.

[Lecturer's experiences in higher education classroom practice](#)

The unit has been implemented three times (in three subsequent years) in exercise classes accompanying an introductory lecture on didactics of mathematics for the first year. Each year, the lecture is attended by around 130 students and has 6–7 exercise courses.

There have been different versions of the unit:

- A) Exercise material and task instruction (covering mathematical and didactical task) were given to a group of 2–4 students who should prepare a 30-minute teaching unit to be held by the group in class (10–30 people) in which they present their solution. The group had a minimum of 2 weeks to prepare. During that time, there were opportunities to talk to the lecturers (office hours, e-mail) and ask questions about the task.

Experiences with A): Many students tried to deal with the task by themselves, so some groups had mathematically incorrect solutions or admitted they couldn't find a solution. In such cases, the lecturers moderated a joint problem-solving exercise in class. It was important for the lecturers that

the class get to know the correct mathematical solution. If there was additional time, some of the above-mentioned topics (didactical or mathematical) were discussed in the remaining time. Unfortunately, past implementations allocated only 45 minutes to an exercise session. We strongly recommend to plan this exercise for 90 minutes, and to maybe prepare discussion material in advance, so the topic can be treated thoroughly and systematically.

- B) Exercise material and task instruction (covering mathematical and didactical task) were given to single students, who were expected to write a 3-page text, summarizing the solution to the mathematical task and giving their own ideas in response to the didactical task. These students were expected to attend the exercise sessions where the topic was discussed and to hand in their text one week after the respective session.

Experiences with B): Some students apparently did not attend any exercise session as a few gave incorrect solutions. However, overall, many students explained the correct solution in their text – more than succeeded to explain it in the presentations. Letting students hand in a text will probably result in many productive activities on the side of the students. However, due to the imposed page-limit of the text and lack of guidance by the lecturer, there were no real discussions in the texts of any of the didactical points mentioned above. Student texts (understandably) focussed on what is asked for in the task: mathematical solution and ideas for application in school.

- C) [Fully digital implementation due to corona in summer 2020:] A task instruction (covering mathematical and didactical task) was given to students. Pre-made groups of students were asked to create digital learning material about the topic (analogue to A) and some individual students were asked to hand in a text of 3–5 pages (analogue to B).

Experiences with C): For questions concerning the implementation of the digitalized course in the summers of 2020 and 2021 please contact a member of the author team. The implementation is constantly modified to adapt to new demands and might be changed again in Summer 2022.

Students with special needs

No special resources have been included for students with special needs.

Assessment

Two questions were used in past exams to assess knowledge of the mathematical solution of the unit. We would evaluate the outcomes for these questions as rather poor (compared to other questions in the exams). The two questions are:

Question 1) – single choice

Three definitions from the schoolbook were given, which were relevant to the notion of inflection point. Furthermore, the drawing of graph 1 from the exercise material was given. The task was, to tick the correct one of four statements about the depicted graph:

1. The graph does not have any inflection points. [correct option]
2. The graph has exactly one inflection point.
3. The graph has exactly two inflection points.
4. The graph has infinitely many inflection points.

Of about 90 participants of a 2019 exam, about 49% ticked the correct statement 1, 19% ticked statement 2, 9% ticked 3 and 23% ticked 4.

Question 2) – multiple-choice

The definition of (local) maximum and minimum from the schoolbook was given. (It is common knowledge that both maximum and minimum are subsumed under the term extremum.)

Furthermore, the drawing of graph 2 from the exercise material was given. The task was, to tick all correct statements out of a selection of five statements about the depicted graph:

1. The graph has no extremum.
2. The graph has exactly one extrema.
3. The graph has exactly two extrema.
4. The graph has infinitely many extrema. [correct option]
5. The graph has infinitely many local maxima. [correct option]

Of about 35 participants of a different 2019 exam, about 56% ticked item 4 correctly, but only about 12% achieved perfect score on this question by also ticking item 5. Items 1, 2 and 3 were each ticked incorrectly by 10–20% of participants.

Relevance of/to the real world

As curve sketching is quite a large topic in German upper high school, we deem this unit, its activities and the questions that can be raised very important for teacher professionalization.

B. Student learning activities

Due to the openness of the task and because we did not systematically observe our large student group, we can describe an array of reactions we have observed/experienced, but cannot give a systematic (statistical) or in any way complete account of what has happened in our different groups. For descriptions of empirical observation of students' learning activities, please contact the authors of this contribution.

Worksheet and files

- The task instruction is given in the file
“[LUH_CurveSketching_UnitMaterial_TaskInstruction.pdf](#)”
- The schoolbook excerpt given to students was:
Freudigmann, H., Buck, H., Greulich, D., Sandmann, R., & Zinser, M. (2012). *Lambacher Schweizer – Mathematik für Gymnasien. Analysis Leistungskurs*. Stuttgart: Ernst Klett Verlag. 38–67